Solution to Assignment 5

45. Use cylindrical coordinates. The volume is

$$\int_{-\pi/2}^{0}\int_{0}^{3\cos\theta}\int_{0}^{-r\sin\theta} r\,dzdrd\theta = \cdots = 9/4\ .$$

59. The two surfaces meet at the unit circle $x^2+y^2=1$. Use cylindrical coordinates. The volume is

$$\int_{0}^{2\pi}\int_{0}^{1}\int_{4r^2}^{5-r^2} r\,dzdrd\theta = \cdots = 5\pi/2\ .$$

Supplementary Problems

1. Let Ω be the bullet-shaped solid bounded above by the upper side of $x^2+y^2+(z-2)^2=1$, below by the xy-plane, and the cylinder $x^2+y^2=1$ on the side. Express the triple integral of a function f over Ω in two ways: $d\rho d\varphi d\theta$ and $dzdrd\theta$.

   Solution. The upper side of the sphere $x^2+y^2+(z-2)^2=1$ is described by $\rho_1 = 2\cos\varphi+\sqrt{4\cos^2\varphi+3}$. The side cylinder is described by $\rho_2 = 1/\sin\varphi$. The region Ω is decomposed to the union of Ω_1 and Ω_2 where

   $$\Omega_1 : 0\le\rho\le\rho_1,\quad 0\le\varphi\le\varphi_0,\quad 0\le\theta\le 2\pi.$$

   Here φ_0 satisfies $\tan\varphi_0 = 1/2$. And

   $$\Omega_2 : 0\le\rho\le 1/\sin\varphi,\quad \varphi_0\le\varphi\le\pi/2,\quad 0\le\theta\le 2\pi.$$

   Hence

   $$\begin{aligned}\iiint_\Omega f\,dV &= \int_0^{2\pi}\int_0^{\varphi_0}\int_0^{\rho_1} f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta + \\ &\quad \int_0^{2\pi}\int_{\varphi_0}^{\pi/2}\int_0^{1/\sin\varphi} f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi\,d\rho d\varphi d\theta.\end{aligned}$$

   In cylindrical coordinates,

   $$\Omega : 0\le 0\le 2+\sqrt{1-r^2},\quad 0\le r\le 1,\quad 0\le\theta\le 2\pi.$$

   Hence

   $$\iiint_\Omega f\,dV = \int_0^{2\pi}\int_0^1\int_0^{2+\sqrt{1-r^2}} f(r\cos\theta, r\sin\theta, z)r\,dzdrd\theta.$$

2. Let D be a region in the plane which is symmetric with respect to the origin, that is, $(x,y)\in D$ if and only if $(-x,-y)\in D$. Show that

   $$\iint_D f(x,y)\,dA(x,y) = 0\ ,$$

   when f is odd, that is, $f(-x,-y) = -f(x,y)$ in D. Suggestion: Convert to polar coordinates.

Solution. Let $\tilde{f}$ be the universal extension of f. It is readily checked that $\tilde{f}$ is an odd function in the entire plane. Let D_1 be a large disk of radius R centered at the origin containing D. By converting to polar coordinates,

$$\begin{aligned} \iint_D f &= \iint_{D_1} \tilde{f}\, dA \\ &= \int_0^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta \\ &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta + \int_{\pi}^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta\ . \end{aligned}$$

Further, using the change of variables $\alpha = \theta - \pi$, the second integral becomes

$$\begin{aligned} \int_{\pi}^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos(\alpha+\pi), r\sin(\alpha+\pi)) r\, dr d\alpha \\ &= \int_0^{\pi} \int_0^R \tilde{f}(-r\cos\alpha, -r\sin\alpha) r\, dr d\alpha \\ &= -\int_0^{\pi} \int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r\, dr d\alpha. \end{aligned}$$

It follows that

$$\begin{aligned} \iint_D f &= \iint_{D_1} \tilde{f}\, dA \\ &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta + \int_{\pi}^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta \\ &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r\, dr d\theta - \int_0^{\pi} \int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r\, dr d\alpha = 0\ . \end{aligned}$$